

Comparison of two reliability models for performance degradation analysis

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Abstract: In order to investigate the general reliability assessment methods based on performance degradation data, two commonly used stochastic process approaches, bilinear process method and random-effect model were studied. Analyzing procedure and effectiveness of these two methodologies were studied and compared. Meanwhile, the two approaches were illustrated through practical applications. The residual plots and the 10th percentile curves of the two methods were presented to demonstrate the comparative results. The random-effect model yielded more volatile residuals and a lower and unsafe 10th percentile curve. Consequently the bilinear process model can be concluded to derive more reasonable results due to its one-stage estimation property.

Key words: degradation analysis; reliability assessment; stochastic process;
bilinear independent increment process; random-effect model

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Products and devices with high reliability are designed to function with a good performance for a long time before they fail. Traditional methods, based on lifetime data, usually face numerous challenges when they are used to assess the reliability status for these high reliable products. The reason is that few or even no failure data can be obtained through general life tests. Even accelerated life tests (ALT) can not relieve this circumstance in a short period, partly because ALT cannot provide enough failures yet and it may cause more errors with the use of extrapolation.

By analyzing the causes of the failure, researchers find that a product's performance generally changes as a degradation process. Nair pointed that the performance degradation data contains abundant reliability information^[1]. And the degradation process provides more information than lifetime data, and this information can

be used to assess product reliability. The relationship between performance degradation and reliability index can be established by the distribution function of degradation performance.

Much attention and many degradation analysis approaches have been established in the past few decades. Chao reviewed some scattered works relating to this field^[2]. The degradation path method illustrated a simple and convenient model for degradation analysis, and this kind of method had been widely used in early time^[3-7]. However, this model has obvious shortcomings because it ignores product's degradation mechanism and the extrapolation of degradation path will strongly affect the accuracy of analysis.

In recently years, methods based on stochastic process have been more and more popular. A typical model of this category is based on the theory of Wiener process^[8-10]. Wang^[11] established a model based on Wiener process

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with random drift. Another commonly used model of this kind is called Gamma process model^[12-13]. The two models studied in the current paper also fall in the scope of stochastic process approach.

According to the studies in Ref. [14] and Ref. [15], estimators $\mu(t)$ and $\sigma(t)$ were obtained to describe the distribution properties of performance measurements at each test time, where $\mu(t)$ is the mean function of the degradation process and $\sigma(t)$ is the standard deviation function. Both $\mu(t)$ and $\sigma(t)$ have linear relationships with degradation time. It is very common in practical degradation test that the degradation process have a linear mean function and a quadratic variance function of transformed time, and the standard deviation function can be further represented by a linear form. Bilinear process method and random-effect model are commonly used to process degradation data exhibiting these statistical characteristics. Because of material property change, manufacturing process variation, stress variation, and many other uncontrollable factors, the random-effect model considers that the values of some or all of the model parameters vary from unit to unit, and they can be drawn randomly from the respective populations^[16]. The bilinear process method is a degradation analysis approach of independent increment process with linear mean function and linear standard deviation function of transformed time^[17]. In Ref. [18], the circumstance in which performance degradation test begins at the initial time $t_0=0$ was discussed, and the standard deviation function is linear with zero intercept, i. e., $\sigma(t)=dt$.

In this paper, analyzing procedure and effectiveness of the two general stochastic process methods, random-effect model and bilinear process model were studied. The rest of this paper is organized as follows. Section 1 introduces the main procedure of bilinear independent increment process model and the random-effect model. An approximately accurate multivariate approach for random-effect model is also discussed

in Section 1. Section 2 illustrates the empirical study and comparative results. Finally, some concluding remarks are given in Section 3.

1 The two approaches

1.1 Bilinear process model

Bilinear process is a special circumstance of general linear independent increment process. It is commonly seen in practical engineering, and also illustrates the property of simplicity. Its mean and standard deviation is taken as linear functions of test time or transformed test time if a transformation is needed before model establishment. In the following descriptions, the test time and degradation measurements are the critical values or transformed ones.

Suppose that a degradation test uses a sample of m units. During testing, each unit is inspected periodically to measure the critical performance characteristic y . Let y_{ij} denote the measurement of y on unit i at time x_{ij} , where $i=1,2,\dots,m$, $j=1,2,\dots,n_i$, and n_i is the number of measurements on unit i . It is assumed that the performance increments $y_2 - y_1, y_3 - y_2, \dots, y_{m_i} - y_{i(n_i-1)}$ are independent of each other for all items $i=1,2,\dots,m$. Increment of time interval $x_{ij} - x_{i(j-1)}$ for item i further follow a normal distribution

$$y_{ij} - y_{i(j-1)} \sim N[b(x_{ij} - x_{i(j-1)}),$$

$$2d_1 d_2 (x_{ij} - x_{i(j-1)}) + d_2^2 (x_{ij}^2 - x_{i(j-1)}^2)] \quad (1)$$

where b is the constant mean degradation rate; d_1 and d_2 are constants depicting the increasing rate of standard deviation.

Based on the properties of independent increment process and the assumption mentioned above, y_{ij} also follows a normal distribution. The statistical model for the i th unit can be expressed by

$$y_{ij} = a + bx_{ij} + \epsilon(x_{ij}) \quad (2)$$

$$\epsilon(x_{ij}) \sim N[0, (d_1 + d_2 x_{ij})^2] \quad (3)$$

$$\text{Cov}[\epsilon(x_{ij}), \epsilon(x_{ik})] = (d_1 + d_2 x_{ij})^2$$

$$x_{ij} \leq x_{ik} \quad (4)$$

where a is unknown model parameters.

For the convenience of estimate and

inference, define $\Delta y_{ij} = y_{ij} - y_{i(j-1)}$, $\Delta x_{ij} = x_{ij} - x_{i(j-1)}$, and $\bar{x}_{ij} = [x_{ij} + x_{i(j-1)}]/2$ relating to the increment of transformed degradation measurement and test time, $i=1, 2, \dots, m$, $j=2, 3, \dots, n_i$. Further define $\sigma_0^2 = d_1 + 2d_2\bar{x}$, and $I(x) = 1 + \theta(x - \bar{x})$, where $\bar{x} = \left[\sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij} \right] / n$ is the overall mean of all transformed times, $n = \sum_{i=1}^m n_i$ is the total number of measurements for all items, and $\theta = 2d_2 / (d_1 + 2d_2\bar{x})$ is a converted parameter that relates to the changes of variance.

With these definitions, the statistical degradation model for the i th unit can be rewritten as

$$y_{i1} \sim N(a + bx_1, \sigma_1^2) \quad (5)$$

$$\Delta y_{ij} \sim N(b\Delta x_{ij}, \sigma_0^2 \Delta x_{ij} I(\bar{x}_{ij})) \quad (6)$$

where $\sigma_0^2 \Delta x_{ij} I(\bar{x}_{ij})$ is the variance increment for transformed degradation inspection increment Δy_{ij} . Then the likelihood function is

$$L = \prod_{i=1}^m \left\{ \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(y_{i1} - a - bx_1)^2}{2\sigma_1^2}\right] \cdot \prod_{j=2}^{n_i} \frac{1}{\sigma_0 \sqrt{2\pi\Delta x_{ij} I(\bar{x}_{ij})}} \exp\left[-\frac{(\Delta y_{ij} - b\Delta x_{ij})^2}{2\sigma_0^2 \Delta x_{ij} I(\bar{x}_{ij})}\right] \right\} \quad (7)$$

The log likelihood function can be written as

$$\begin{aligned} \ln L = & -\frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum_{i=1}^m \sum_{j=2}^{n_i} \ln[\Delta x_{ij} I(\bar{x}_{ij})] - \\ & \frac{m}{2} \ln \sigma_1^2 - \frac{1}{2\sigma_0^2} \sum_{i=1}^m \sum_{j=2}^{n_i} \frac{(\Delta y_{ij} - b\Delta x_{ij})^2}{\Delta x_{ij} I(\bar{x}_{ij})} - \\ & \frac{n-m}{2} \ln \sigma_0^2 - \frac{1}{2\sigma_1^2} \sum_{i=1}^m (y_{i1} - a - bx_1)^2 \end{aligned} \quad (8)$$

Model parameters can be determined by solving the following equations

$$\frac{\partial}{\partial a} \ln L = \frac{1}{\sigma_1^2} \sum_{i=1}^m (y_{i1} - a - bx_1) = 0 \quad (9)$$

$$\begin{aligned} \frac{\partial}{\partial b} \ln L = & \frac{1}{\sigma_1^2} \sum_{i=1}^m (y_{i1} - a - bx_1) x_1 + \\ & \frac{1}{\sigma_0^2} \sum_{i=1}^m \sum_{j=2}^{n_i} \frac{\Delta y_{ij} - b\Delta x_{ij}}{I(\bar{x}_{ij})} = 0 \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{\partial}{\partial \sigma_0^2} \ln L = & -\frac{n-m}{2\sigma_0^2} + \\ & \frac{1}{2\sigma_0^4} \sum_{i=1}^m \sum_{j=2}^{n_i} \frac{(\Delta y_{ij} - b\Delta x_{ij})^2}{\Delta x_{ij} I(\bar{x}_{ij})} = 0 \end{aligned} \quad (11)$$

$$\frac{\partial}{\partial \sigma_1^2} \ln L = -\frac{m}{2\sigma_1^2} +$$

$$\frac{1}{2\sigma_1^4} \sum_{i=1}^m (y_{i1} - a - bx_1)^2 = 0 \quad (12)$$

Setting θ to be fixed, the conditional maximum likelihood estimates of parameters a, b, σ_0^2 and σ_1^2 can be derived. Then, substituting these estimates into the log likelihood function, Eq. (8), and simplify. The profile log likelihood function can be obtained

$$\begin{aligned} \ln L = & -\frac{n}{2} \ln(2\pi) - \frac{n-2}{2} - \frac{m}{2} \ln \sigma_1^2 - \\ & \frac{n-m}{2} \ln \hat{\sigma}_0^2(\theta) - \frac{1}{2} \sum_{i=1}^m \sum_{j=2}^{n_i} \ln[\Delta x_{ij} I(\bar{x}_{ij})] \end{aligned} \quad (13)$$

Therefore, the maximum likelihood estimate $\hat{\theta}$ can be obtained by maximizing the profile log likelihood function in Eq. (12) through a binary search. The parameter determination procedure and expressions are given in Ref. [17]. Returning to the original statistical inferences in degradation analysis, the performance at some future time x can be predicted by

$$\hat{y} = \hat{a} + \hat{b}x \quad (14)$$

and the estimate of the corresponding variance can be obtained by

$$\begin{aligned} \hat{\sigma}^2(x) = & \hat{\sigma}_1^2 + (x - x_1) I\left(\hat{\theta}, \frac{x + x_1}{2}\right) \hat{\sigma}_0^2 = \\ & \hat{d}_0 + \hat{d}_1 x + \hat{d}_2 x^2 \end{aligned} \quad (15)$$

Considering the statistical inference for the performance depicted by Eq. (14), keeping θ fixed, it can also be given by

$$\hat{y} = \hat{y}_1 + \hat{b}(x - x_1) \quad (16)$$

Keeping θ fixed, then it can be concluded that

$$\hat{y} \sim N\left[a + bx, \frac{\sigma_1^2}{m} + (x - x_1)^2 \sigma_0^2 / \sum_{i=1}^m \sum_{j=2}^{n_i} \frac{\Delta x_{ij}}{I(\bar{x}_{ij})}\right] \quad (17)$$

Consequently, its confidence interval can be derived.

1.2 Random-effect model

Also let y_{ij} denote the measurement for the i th item at test time x_{ij} , then the degradation path can be modeled by

$$y_{ij} = g(x_{ij}; \beta_{1i}, \beta_{2i}, \dots, \beta_{pi}) + e_{ij} \quad (18)$$

where $g(x_{ij}; \beta_{1i}, \beta_{2i}, \dots, \beta_{pi})$ is the true degradation of y for item i at time x_{ij} and e_{ij} is the error term.

It can be seen that the values of the degradation model parameters $\beta_1, \beta_2, \dots, \beta_p$ vary from unit to unit. They are not independent; they usually form a joint distribution. A multivariate normal distribution with mean vector $\boldsymbol{\mu}_\beta$ and variance-covariance matrix $\boldsymbol{\Sigma}_\beta$ is often used for simplicity. By directly maximizing the likelihood function, the model parameters can be derived. However, the calculation procedure is an extremely difficult process unless the true degradation path takes a simple form^[17].

In this paper, the approximate but simple multivariate approach for model parameter estimation is adopted. (1) Fit the degradation model, depicted by Eq. (18), to each individual degradation path, and obtain the parameter estimates $\hat{\beta}_{1i}, \hat{\beta}_{2i}, \dots, \hat{\beta}_{pi}$ by the least square method for each unit $i=1, 2, \dots, m$. (2) Consider the estimates of each parameter as a sample of m observations. Then the sample mean vector can be expressed by

$$\bar{\boldsymbol{\beta}} = [\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_p] \quad (19)$$

where

$$\hat{\beta}_j = \frac{1}{m} \sum_{i=1}^m \hat{\beta}_{ji} \quad j = 1, 2, \dots, p \quad (20)$$

And the sample variance covariance matrix can be calculated by

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & s_{22} & \cdots & s_{2p} \\ \vdots & \vdots & & \vdots \\ s_{p1} & s_{p2} & \cdots & s_{pp} \end{bmatrix} \quad (21)$$

where s_{kj} , $k \neq j$, is the covariance of β_k and β_j , and s_{kk} is the sample variance of β_k . And it can be determined by

$$s_{kj} = \frac{1}{m} \sum_{i=1}^m (\hat{\beta}_{ki} - \hat{\beta}_k)(\hat{\beta}_{ji} - \hat{\beta}_j) \quad k, j = 1, 2, \dots, p \quad (22)$$

In order to obtain the unbiased estimate, the variance-covariance component is corrected by replacing m with $m-1$ in Eq. (22), when the

sample size is considerably small, say $m < 15$. In addition, the association between model parameter β_k and β_j can be described by the correlation coefficient ρ_{kj} , which is defined as

$$\rho_{kj} = \frac{s_{kj}}{\sqrt{s_{kk}} \sqrt{s_{jj}}} \quad k, j = 1, 2, \dots, p \quad (23)$$

Then, performance and its corresponding variance at some future test time can be obtained. And the performance confidence interval can accordingly be calculated.

2 Empirical comparative result

In this section, the practical performance of bilinear process method and random-effect model is focused on. A self-regulating heating cable was produced in California with extensive application in conditions that require high reliability. Extensive degradation test was conducted, both to ensure the dependability of current products and to support the development of new and improved product designs.

Degradation of the cable is indicated by its electrical resistance R increase with age. In the degradation test conducted in Ref. [19], the specimens were sample cable lengths cut from a production lot. Accordingly, the natural log of resistance are also considered as the degradation measurement. Figure 1 shows the natural log of resistance of 5 test items measured at 200 °C. Each unit is measured at the same 10 predefined test times, which are not equally spaced in the whole period of 5.608×10^3 h.

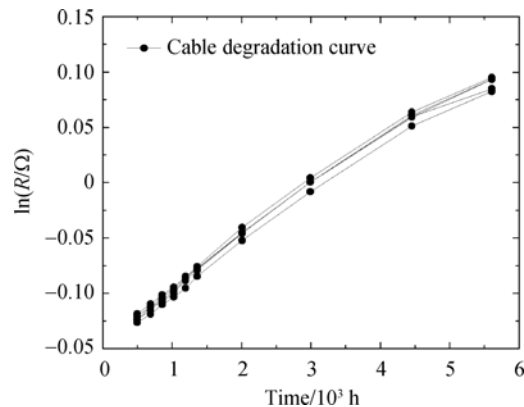


Fig. 1 Natural log of resistance for heating cable

A preliminary statistical analysis, similar with the procedure adopted by Yang and Xue^[14], has been conducted on the cable degradation data, and the values of mean and standard deviation at each observational time were derived. Based on these results, it can be inferred that the mean and the standard deviation of the degradation process both vary linearly with test time. Consequently, the bilinear process method and random-effect model discussed in Section 1 can properly be utilized to process this data set.

According to the bilinear process method, the following regression model can be derived to depict the cable degradation process.

$$\hat{y}(x) = -0.1453 + 4.5355 \times 10^{-2}x \quad (24)$$

$$\hat{\sigma}(x) = 7.099 \times 10^{-4} + 3.5634 \times 10^{-3}x \quad (25)$$

With the random-effect approach, theregression equation

$$\hat{y}^*(x) = -0.1396 + 4.3067 \times 10^{-2}x \quad (26)$$

the estimate of the variance-covariance matrix

$$\hat{\mathbf{S}} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 1.0191 \times 10^{-5} & 6.8244 \times 10^{-7} \\ 6.8244 \times 10^{-7} & 3.6634 \times 10^{-7} \end{bmatrix} \quad (27)$$

and the correlation coefficient

$$\hat{\rho}_{12} = \frac{s_{12}}{\sqrt{s_{11}} \sqrt{s_{22}}} = 6.8245 \times 10^{-7} \quad (28)$$

can all be obtained. Accordingly, the variance estimation can be calculated as

$$\hat{\sigma}^{*2}(x) = 1.0191 \times 10^{-5} + 1.3649 \times 10^{-6}x + 3.6634 \times 10^{-7}x^2 \quad (29)$$

To demonstrate the rationality of the modeling process for the two approaches, the residual plots over time are shown in Fig. 2 and Fig. 3, respectively. Accordingly, it can be concluded that the bilinear process method is more appropriate to describe the heating cable degradation data set.

Furthermore, comparative results of the 10th percentile curve $\hat{y}_{0.1}(x)$ for the heating cable degradation are illustrated in Fig. 4. According to the concept, it can be concluded that 90% of the population of the heating cable perform-

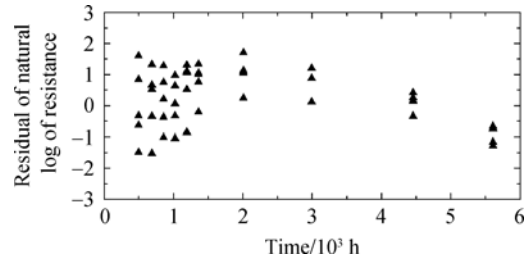


Fig. 2 Residual of natural log of resistance plot from bilinear process method

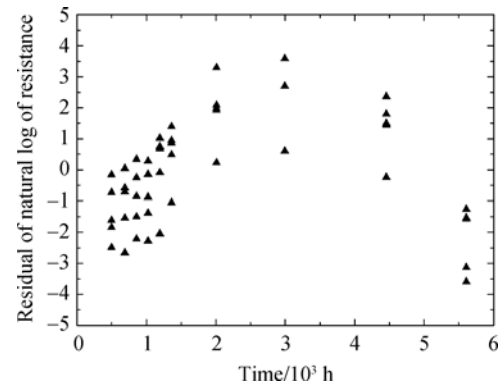


Fig. 3 Residual of natural log of resistance plot from random-effect model

ance $y(x)$ will be smaller than $\hat{y}_{0.1}(x)$.

From Fig. 4, it can be seen that most test data is under the 10th percentile curve derived by the bilinear process method, while much test data is above that from the random-effect model.

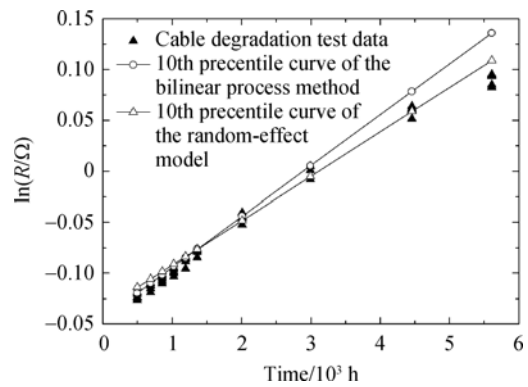


Fig. 4 10th percentile curve for heating cable degradation

3 Conclusions

Modeling and analyzing performance degradation data has shown clear and significant sense for complex systems and highly reliable

products. The reason is that enough failure life data through a relatively short period of time available to conduct the internal life test is hardly to get. Therefore the traditional reliability theory, for which ample failure life data is essential to ensure the precision, usually encounters numerous challenges.

Two commonly used performance degradation modeling approaches, bilinear process method and random-effect model, were studied in the current paper. Theoretical analysis and empirical results suggested the reasonability and effectiveness of the bilinear process approach, and showed a promise for future applications in degradation analysis.

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